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Equilibrium conditions for coupled classical–quantum systems

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Abstract. Systems with classical as well as quantal degrees of freedom are described. This is done by means of a relative Hamiltonian and a derivation satisfying a compatibility relation. Various equivalent equilibrium conditions are studied.

1. Introduction

Various rigorous definitions of thermodynamical equilibrium have been studied for about two decades. All these definitions avoid the thermodynamical limit and are immediately expressed in terms of conditions on the equilibrium states for the infinite system.

The Dobrushin-Lanford-Ruelle equations were introduced for classical lattice systems [1, 2] and the classical Kubo-Martin-Schwinger condition [3] for continuous classical systems equipped with a Poisson bracket structure. The DLR equations were later expressed [4, 5] in terms of relative Hamiltonians.

Parallel to these classical equilibrium conditions, one has the quantum KMS condition [6]. These conditions existed and were applied in various problems and reformulated in different ways. A main aspect of the other formulations is that the KMS conditions are determined by a derivation.

Various generalisations of both the DLR equations [7] and the KMS conditions [8] have been studied. Here we mention in particular those generalisations which had in mind a rephrasing of the KMS and DLR equations such that they obey one and the same principle [8-10]. An open problem is the study of the equilibrium condition for a coupled interacting system composed of a quantum and a classical system, or a quantum system with also classical degrees of freedom or vice versa. The object of this article is to develop the theoretical basis and the equilibrium conditions for the states of such systems. Recent examples of the type of systems we are studying are found in [11] and [12].

2. Preliminaries

Consider $\mathscr{X}$ the configuration space of the classical variables which we take to be a compact Hausdorff space for technical convenience and take $\mathscr{A}$ to be a simple C^* algebra describing the quantum degrees of freedom. The algebra of observables for

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the composed system is taken to be the C^* algebra $\mathscr{B} = \mathscr{C}(\mathscr{K}) \otimes \mathscr{A}$ of continuous functions $A: x \in \mathscr{K} \to A(x) \in \mathscr{A}$ from $\mathscr{K}$ into $\mathscr{A}$ equipped with the norm

$$\|A\| = \sup_{x \in \mathscr{K}} \|A(x)\|.$$

Denote by $\mathscr{Q}$ a group of homeomorphisms of the configuration space $\mathscr{K}$ acting in such a way that $\mathscr{Q}$ separates the continuous functions $\mathscr{C}(\mathscr{K})$, i.e. the only invariant functions are the constant ones. For any $\tau \in \mathscr{Q}$, define $\tilde{\tau}$ the induced action on $\mathscr{B}$ by

$$(\tilde{\tau} A)(x) = A(\tau^{-1} x) \qquad x \in \mathscr{K},\ A \in \mathscr{B}.$$

For classical systems the equilibrium states are determined by specifying a relative Hamiltonian. For quantum systems they are determined by a derivation describing the interactions. Here we treat composed systems. Therefore we should also include interactions between the classical and quantum degrees of freedom. Hence we suppose that we are given a derivation δ defined on a dense $\mathscr{Q}$-invariant domain $\mathscr{B}_0$ satisfying

$$\begin{aligned} &\delta(A)^* = -\delta(A^*) \\ &\delta(AB) = \delta(A)B + A\delta(B) \qquad A, B \in \mathscr{B} \end{aligned} \tag{1}$$

and a relative Hamiltonian h, i.e. a map h of $\mathscr{Q}$ into $\mathscr{B}$,

$$h: \tau \in \mathscr{Q} \to h(\tau) \in \mathscr{B}_0$$

satisfying

$$\begin{aligned} &h(\tau)^* = h(\tau) \\ &h(\tau_1 \tau_2) = \tilde{\tau}_1 h(\tau_2) + h(\tau_1) \qquad \tau_1, \tau_2 \in \mathscr{Q}. \end{aligned} \tag{2}$$

Furthermore h and δ are connected to each other by the relation

$$\tilde{\tau}\delta\tilde{\tau}^{-1} - \delta = [h(\tau), \cdot\,] \qquad \tau \in \mathscr{Q}. \tag{3}$$

To fix our ideas we give the example of a Hamiltonian system on a lattice $\mathbb{Z}^\nu$. We take $\mathscr{K} = \bigtimes K_{j \in \mathbb{Z}^\nu}$ with $K = \{1, \ldots, n\}$ and $\mathscr{A}$ the spin UHF algebra: $\mathscr{A} = \otimes_{i \in \mathbb{Z}^\nu} M_n$ with M_m the $m \times m$ complex matrices. An interaction is a set $\phi = \{(\phi_\Lambda | \Lambda \subset \mathbb{Z}^\nu)\}$ of elements ϕ_Λ of $\mathscr{B}$.

Suppose that the interaction is of finite range and assume

$$\sup_\Lambda \|\phi_\Lambda\| < \infty.$$

The local Hamiltonians are then given by

$$H_\Lambda = \sum_{\Lambda' \in \Lambda} \phi_{\Lambda'}. \tag{4}$$

In this case the derivation δ is given by

$$\delta(A) = \lim_{\Lambda \to \mathbb{Z}^\nu} [H_\Lambda, A] \tag{5}$$

where A is any local observable. In order to define the relative Hamiltonian we take for $\mathscr{Q}$ the set of local transformations of $\mathscr{K} = \bigtimes K_{j \in \mathbb{Z}^\nu}$, i.e. τ is a local transformation if there exists a finite Λ such that $\mathscr{K}_{\Lambda^0} = \bigtimes K_{j \notin \Lambda}$ is left invariant under τ.

The relative Hamiltonian is then

$$h(\tau) = \lim_{\Lambda \to \mathbb{Z}^\nu} (\tilde{\tau} H_\Lambda - H_\Lambda). \tag{6}$$

It is easily checked that the relations (1)–(3) hold.

The static Hubbard model, recently studied [11], is an example of these Hamiltonian systems described above. The local Hamiltonians are given by

$$H_\Lambda = \sum_{i,j\in\Lambda} t(i-j)a_i^+ a_j + 2U \sum_{i\in\Lambda} a_i^+ a_i W(i)$$

where the a_i and a_i^+ are the Fermion creation and annihilation operators and $W(i)$ takes the values 0 or 1.

We remark also that it is possible to reconstruct local Hamiltonians from a given derivation and a given relative Hamiltonian satisfying (1)-(3).

3. Finite systems

We refer to a finite system if the algebra of observables $\mathscr{B}$ is constructed by means of a configuration space $\mathscr{X} = \{1, \ldots, n\}$ consisting of a finite number of points and the algebra $\mathscr{A} = M_m$, the set of $m \times m$ complex matrices. A state ω of $\mathscr{B}$ is then described by a density matrix $\rho \in \mathscr{B}$ such that

$$\omega(A) = \sum_{x\in\mathscr{X}} \mathrm{Tr}(\rho(x)A(x)) \qquad A \in \mathscr{B}$$

with $\rho(x) \geqslant 0$

$$\sum_x \mathrm{Tr}(\rho(x)) = 1.$$

A Hamiltonian is a self-adjoint element $H^* = H$ of $\mathscr{B}$, i.e. for all $x \in \mathscr{X}: H(x)^* = H(x)$.

As usual a canonical Gibbs state with Hamiltonian H at inverse temperature β is given by the state ω_β of $\mathscr{B}$

$$\omega_\beta(A) = \left(\sum_x \mathrm{Tr}[A(x)\exp(-\beta H(x))]\right)\left(\sum_x \mathrm{Tr}[\exp(-\beta H(x))]\right)^{-1} \qquad A \in B. \tag{7}$$

As in (5) and (6) we denote

$$\delta = [H, \cdot\,]$$

$$h(\tau) = \tilde{\tau}H - H$$

and then have the following theorem.

Theorem 3.1. Let ω be a state of the finite system with Hamiltonian H, then the following are equivalent:

(i) ω is a Gibbs state for H at inverse temperature β,

(ii) ω satisfies the equilibrium conditions (EC) equation that for all $\tau \in \mathscr{Q}$ and $V \in \mathscr{B}$ one has

$$\omega(\tilde{\tau}(VV^*)) = \omega(V^*[\exp(-\beta\Gamma_\tau)]V) \tag{8}$$

where $\Gamma_\tau(A) = h(\tau^{-1})A + \delta(A)$, $A \in \mathscr{B}$, and

(iii) ω satisfies the entropy-energy balance (EEB) inequality that for all $\tau \in \mathscr{Q}$ and $V \in \mathscr{B}$ one has

$$\beta\omega[V^*(h(\tau^{-1})V + \delta(V))] \geqslant \omega(V^*V)\ln[\omega(V^*V)/\omega(\tilde{\tau}(VV^*))]. \tag{9}$$

Proof. Suppose first that $\omega=\omega_\beta$ (see (7)), then for all $V\in\mathscr{B}$ and $\tau\in\mathscr{Q}$:

$$\begin{aligned}\omega_\beta(\tilde{\tau}(VV^*)) &= \frac{1}{Z}\sum_x \mathrm{Tr}[\exp(-\beta H(x))\,V(\tau^{-1}(x))\,V(\tau^{-1}(x))^*]\\ &= \frac{1}{Z}\sum_x \mathrm{Tr}[\,V(x)^*\exp(-\beta H(\tau x))\,V(x)]\\ &= \frac{1}{Z}\sum_x \mathrm{Tr}[\exp(-\beta H(x))\,V(x)^*\exp(-\beta H(\tau x))\,V(x)\exp(\beta H(x))]\\ &= \omega[\,V^*\exp(-\beta\Gamma_\tau)\,V]\end{aligned}$$

where $Z=\Sigma_x\,\mathrm{Tr}[\exp(-\beta H(x))]$ and Γ_τ is the map of $\mathscr{B}$ defined by

$$\Gamma_\tau(A)(x)=H(\tau x)A(x)-A(x)H(X)\qquad A\in\mathscr{B}.$$

Hence

$$\Gamma_\tau(A)=h(\tau^{-1})A+\delta(A)$$

and (i) implies (ii).

Suppose now that ω is a state satisfying (ii). For $\tau=$ identity transformation, equation (8) reduces to the KMS condition for the derivation δ, hence $\omega\circ\delta=0$. But this implies that for all $\tau\in\mathscr{Q}$ and $A,B\in\mathscr{B}$:

$$\omega(\Gamma_\tau(A)^*B)=\omega(A^*\Gamma_\tau(B)).$$

This means that Γ_τ defines a self-adjoint operator, which we denote again by Γ_τ on the GNS-Hilbert space $\mathscr{B}$ equipped with the scalar product

$$(A,B)=\omega(A^*B).$$

We remark that in general, Γ_τ is not Hermitian on $\mathscr{B}$. Then applying the Jensen inequality we obtain

$$\begin{aligned}\omega(\tilde{\tau}(VV^*)) &= \frac{\omega(V^*\exp(-\beta\Gamma_\tau)\,V)}{\omega(V^*V)}\,\omega(V^*V)\\ &\geqslant \omega(V^*V)\exp[-\beta\omega(V^*\Gamma_\tau(V))/\omega(V^*V)]\end{aligned}$$

thus proving (iii).

Finally we prove that (iii) implies (i). We remark first that for $V=V^*$, (iii) implies that

$$\omega(V\delta(V))\geqslant 0.$$

Hence

$$\omega(V\delta(V))=\overline{\omega(V\delta(V))}$$

or

$$\omega(\delta(V^2))=0.$$

Therefore for all $x=\mathscr{K}$

$$[H(x),\rho(x)]=0$$

where ρ is the density matrix of the state ω, i.e. there exists an orthonormal basis $\{\phi_i(x) | i = 1, \ldots, m\}$ diagonalising simultaneously $H(x)$ and $\rho(x)$,

$$H(x)\phi_i(x) = \varepsilon_i(x)\phi_i(x)$$

$$\rho(x)\phi_i(x) = \rho_i(x)\phi_i(x).$$

Now we compute the $\rho_i(x)$ from (9) and substitute in the EEB inequality (9):

$$V(x) = \delta_{x_1 x_2} | \phi_i(x_1)\rangle\langle\phi_j(x_2)|$$

$$\tau(x_1) = x_2 \qquad \tau(x_2) = x_1$$

where $\delta_{x,y}$ is the Kronecker δ and where $|\phi_i(x_1)\rangle\langle\phi_j(x_2)|$ is the partial isometry with domain $\phi_j(x_2)$ and image $\phi_i(x_1)$. Then

$$(V^*V)(x) = \delta_{x_1 x_2} | \phi_j(x_2)\rangle\langle\phi_j(x_2)|$$

$$\tilde{\tau}(VV^*)(x) = \delta_{x_1 x_2} | \phi_i(x_1)\rangle\langle\phi_i(x_1)|$$

$$V^*(h(\tau^{-1})V + \delta(V))(x) = \delta_{x_1 x_2}(\varepsilon_i(x_1) - \varepsilon_j(x_2)) | \phi_j(x_2)\rangle\langle\phi_j(x_2)|$$

and from (9)

$$\beta(\varepsilon_i(x_1) - \varepsilon_j(x_2))\rho_j(x_2) \geqslant \rho_j(x_2)\ln(\rho_j(x_2)/\rho_i(x_1)). \qquad (10a)$$

Taking now

$$V(x) = \delta_{x_1 x_2} | \phi_j(x_2)\rangle\langle\phi_i(x_1)|$$

we obtain

$$\beta(\varepsilon_j(x_2) - \varepsilon_i(x_1))\rho_i(x_1) \geqslant \rho_i(x_1)\ln(\rho_i(x_1)/\rho_j(x_2)). \qquad (10b)$$

Suppose $\rho_j(x_2) = 0$, then from the inequality (10*b*) it follows that $\rho_i(x_1) = 0$ and this for all $i = 1, \ldots, n$ and $x_1 \in \mathcal{K}$. Hence $\rho = 0$, but this is contradictory with $\Sigma_x \operatorname{Tr}(\rho(x)) = 1$. Therefore for all $j = 1, \ldots, n$ and $x_2 \in \mathcal{K}$, $\rho_j(x_2) \neq 0$. The inequalities (10*a*) and (10*b*) then yield

$$\rho_j(x_2)/\rho_i(x_1) = \exp[-\beta(\varepsilon_j(x_2) - \varepsilon_i(x_1))]$$

or equivalently

$$\rho_j(x_2)\exp(\beta\varepsilon_j(x_2)) = \rho_i(x_1)\exp(\beta\varepsilon_i(x_1)) = \lambda$$

where the parameter λ is independent of the configurations x and of the indices i, j, and is fixed by the normalisation of the density matrix, yielding

$$\rho = \exp(-\beta H)\left(\sum_x \operatorname{Tr}[\exp(-\beta H(x))]\right)^{-1}.$$

As is well known, an equivalent characterisation of a Gibbs state is given by the variational principle. We denote the free energy of the state ω with density matrix ρ as

$$F(\omega) = \omega(H) + \frac{1}{\beta}\sum_x \operatorname{Tr}[\rho(x)\log(\rho(x))]. \qquad (11)$$

The Gibbs state ω_β is uniquely determined by the minimum principle

$$F(\omega_\beta) \leqslant F(\omega) \qquad \text{for all states } \omega.$$

We now describe an alternative derivation of the EEB inequality, explaining its physical meaning as expressing the balance between energy and entropy increase under a dissipative perturbation of the equilibrium state.

Consider the map $L_{V,\tau}$ of $\mathscr{B}$

$$L_{V,\tau}(A)(x)=V(x)^*A(\tau(x))V(x)-\tfrac{1}{2}(V(x)^*V(x)A(x)+A(x)V(x)^*V(x)).$$

We check that for all $A\in\mathscr{B}$

$$L_{V,\tau}(A^*)A+A^*L_{V,\tau}(A)\leqslant L_{V,\tau}(A^*A).$$

$L_{V,\tau}(\mathbb{1})=0$, therefore $\{\exp(\lambda L_{V,\tau})\,|\,\lambda\in\mathbb{R}^+\}$ is a semigroup of positive unity preserving transformations of $\mathscr{B}$, mapping by duality states into states.

From (11) one has:

$$\lim_{\lambda\to 0^+}\frac{F(\omega_\beta\circ\exp(\lambda L_{V,\tau}))-F(\omega_\beta)}{\lambda}\geqslant 0$$

and by an argument analogous to that in [13] one obtains the EEB inequality (9).

We remark that for infinite systems the EEB inequality (9) remains meaningful, although the variational principle should be reformulated in terms of the free energy density which limits the above derivation to space translation invariant states.

At this point it is instructive to refer to the literature for other unified approaches of the classical and quantum equilibrium conditions. The idea in [8, 9] consists of starting from the quantum mechanical KMS equation in the form of Green functions:

$$\mu_{V^*V}(E)/\nu_{V^*V}(E)=\phi(E)$$

where $\phi(E)=\exp(\beta E)$, $\mu_{V^*V}(E)$ is the Fourier transform of the function $t\to\omega(VV_t^*)$, $\nu_{V^*V}(E)$ the Fourier transform of $t\to\omega(V^*V_t)$ and V_t the time-evolved observables of V. Then other functions ϕ are allowed in order to include what is called the classical KMS condition for systems with a Poisson bracket structure:

$$\beta\omega(A\{H,B\})=\omega(\{A,B\})\qquad A, B \text{ observables}\qquad(12)$$

where $\{\,,\}$ is the Poisson bracket. A distinction is made between classical and quantum mechanical systems by choosing a different function ϕ for the two different situations. This approach does not seem to be suitable for the treatment of coupled classical-quantum systems.

The idea in [10] is to take the point of view of the DLR equation, expressing that locally perturbed equilibrium states are absolutely continuous with respect to the unperturbed one, with a Radon-Nikodym derivative of the local Gibbs factor. The KMS condition is then rewritten in this spirit but without any pretention of treating coupled systems.

It is evident that the EC equations (8) contain both the classical and quantum aspects as expressed by DLR and KMS conditions. The same holds for the EEB inequalities (9). In particular if the system is classical then the algebra $\mathscr{A}$ is reduced to the complex numbers and the EC equation reduces to the DLR equations. If the system is purely a quantum system then the set $\mathscr{K}$ is trivial and $h(\tau)=0$, the EC equation (8) reduces to the KMS equation. Clearly, also the EEB inequalities coincide with the classical or quantum correlation inequalities [14, 15] when the system is purely classical or quantal.

We will avoid in this paper the explicit introduction of systems with a Poisson bracket structure because one excludes in this way the classical lattice systems. The

EC equations and the EEB inequalities, however, are meaningful equilibrium conditions for both lattice systems and systems with a Poisson bracket structure. The equivalence of the classical KMS condition (12) with the DLR equations was studied in [3]. A heuristic argument to deduce equation (12) from the EEB inequalities is

$$\beta\omega(Ah(\tau_s)) \geqslant \omega(A)\ln(\omega(A)/\omega(\tilde{\tau}_s A)) \tag{13}$$

where $\tilde{\tau}_s(A) = \exp(s\{B, \cdot\})A \qquad s \in \mathbb{R}$.

Developing (13) with respect to the parameter s yields

$$\beta s\omega(A\{B, H\}) \geqslant -s\omega(\{B, A\}) + O(s^2)$$

for all values of s, straightforwardly yielding equation (12).

4. Infinite systems

For infinite systems the notion of the Gibbs state loses, strictly speaking, its meaning. However, the notions of the EC equation and the EEB inequality remain meaningful. Here we aim at proving their equivalence also for infinite systems.

We therefore suppose that we are given a derivation δ and a relative Hamiltonian h satisfying the conditions (1)-(3). Moreover, we assume that δ generates a strongly continuous one-parameter group $\{\alpha_t = \exp(it\delta) | t \in \mathbb{R}\}$ of *-automorphisms of $\mathscr{B}$. Denoting for $\tau \in \mathscr{Q}$

$$\Gamma_\tau(A) = \delta(A) + h(\tau^{-1})A \qquad A \in \mathscr{B}$$

we have the following lemma.

Lemma 4.1. For $\tau \in \mathscr{Q}$, the map Γ_τ is exponentiable and defines a strongly continuous one-parameter group $\{\alpha_t^\tau = \exp(it\Gamma_\tau) | t \in \mathbb{R}\}$ of isometries of $\mathscr{B}$.

Proof. Since $h(\tau^{-1}) \in \mathscr{B}$ the exponentiability of Γ_τ follows in a standard way [16] from the Dyson expansion. Indeed, for all $A \in \mathscr{B}$, the series

$$\alpha_t(A) + \sum_{n \geqslant 1} \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n\, \alpha_{t-t_1}(h(\tau^{-1})) \ldots \alpha_{t-t_n}(h(\tau^{-1}))\alpha_t(A)$$

is norm convergent and defines $\exp(it\Gamma_\tau)$ for $t \geqslant 0$ (a similar expansion holds for $t \geqslant 0$). The group property and the strong continuity follow immediately.

In order to prove the isometry property, consider $A \in \mathscr{B}_0$:

$$\begin{aligned}
\frac{1}{i}\frac{d}{dt}&(\alpha_t^\tau(A)\alpha_t^\tau(A)^*) \\
&= \frac{1}{i}\frac{d}{dt}(\alpha_t^\tau(A))\alpha_t^\tau(A)^* + \alpha_t^\tau(A)\frac{1}{i}\frac{d}{dt}(\alpha_t^\tau(A)^*) \\
&= \delta(\alpha_t^\tau(A))\alpha_t^\tau(A)^* + h(\tau^{-1})\alpha_t^\tau(A)\alpha_t^\tau(A)^* \\
&\quad + \alpha_t^\tau(A)\delta(\alpha_t^\tau(A)^*) - \alpha_t^\tau(A)\alpha_t^\tau(A)^* h(\tau^{-1}) \\
&= (\delta + [h(\tau^{-1}), \cdot])(\alpha_t^\tau(A)\alpha_t^\tau(A)^*).
\end{aligned}$$

Using formula (3)

$$\frac{1}{i}\frac{d}{dt}(\alpha_t^\tau(A)\alpha_t^\tau(A)^*) = (\tilde{\tau}^{-1}\delta\tilde{\tau})(\alpha_t^\tau(A)\alpha_t^\tau(A)^*).$$

Hence

$$\alpha_t^\tau(A)\alpha_t^\tau(A)^* = \tilde{\tau}^{-1}\alpha_t\tilde{\tau}(AA^*)$$

and

$$\|\alpha_t^\tau(A)\|^2 = \|\alpha_t^\tau(A)\alpha_t^\tau(A)^*\| = \|AA^*\| = \|A\|^2.$$

Now that we have a strongly continuous one-parameter group $\{\alpha_t^\tau | t \in \mathbb{R}\}$ of isometries of $\mathscr{B}$ for each fixed $\tau \in \mathscr{Q}$ we are in a position to define a dense set $\mathscr{B}_\tau$ of analytic elements for the map Γ_τ (see [16], § 2.5.20).

For all $f \in C_0^\infty(\mathbb{R})$ such that its Fourier transform $\hat{f}$ is of compact support, and for all $A \in \mathscr{B}$, we define

$$A(f) = \int \mathrm{d}t\, f(t)\alpha_t^\tau(A).$$

Take now for $\mathscr{B}_\tau$ the linear span of the $A(f)$.

Clearly

$$\alpha_{\mathrm{i}\beta}^\tau = \exp(-\beta\Gamma_\tau)$$

is well defined on $\mathscr{B}_\tau$.

Definition 4.2.

(i) A state ω of $\mathscr{B}$ satisfies the EC equation at β if for all $\tau \in \mathscr{Q}$ and $A, B \in \mathscr{B}_\tau$, there holds

$$\omega(\tilde{\tau}(AB)) = \omega(B\alpha_{\mathrm{i}\beta}^\tau(A)). \tag{14}$$

(ii) A state ω of $\mathscr{B}$ satisfies the EEB inequality at β if for all $\tau \in \mathscr{Q}$ and $A \in \mathscr{B}_0$, domain of δ, there holds

$$\beta\omega(A^*\Gamma_\tau(A)) \geq \omega(A^*A)\ln(\omega(A^*A)/\omega(\tilde{\tau}(AA^*))). \tag{15}$$

Theorem 4.3. A state ω of $\mathscr{B}$ satisfies the EC equation at β iff it satisfies the EEB inequality at β.

Proof. Suppose first that ω satisfies the EC equation at β, then in particular the EC equation (14) is satisfied for τ the identity map where it reduces to the KMS equation. It follows by the usual argument that the state ω is α_t invariant and separating. Let $(\pi, \Omega, \mathscr{H})$ be the GNS triplet of the state ω and define $\tilde{\Gamma}_\tau$ by

$$\tilde{\Gamma}_\tau(\pi(A)\Omega) = \pi(\Gamma_\tau(A))\Omega \qquad A \in \mathscr{B}.$$

Then, using the α_t invariance of ω,

$$\begin{aligned}(\tilde{\Gamma}_\tau(\pi(A)\Omega), \pi(B)\Omega) &= \omega(\Gamma_\tau(A)^*B) \\ &= \omega((-\delta(A^*) + A^*h(\tau^{-1}))B) \\ &= \omega(A^*(\delta(B) + h(\tau^{-1})B)) \\ &= \omega(A^*\Gamma_\tau(B)) \\ &= (\pi(A)\Omega, \tilde{\Gamma}_\tau(\pi(B)\Omega)).\end{aligned}$$

As $\{\pi(A)\Omega | A \in \mathscr{B}_\tau\}$ is a dense set of analytic vectors, $\tilde{\Gamma}_\tau$ is essentially self-adjoint on $\mathscr{H}$. The EC equation becomes

$$(\pi(\tilde{\tau}A)\Omega, \pi(\tilde{\tau}B)\Omega) = (\pi(B^*)\Omega, \exp(-\beta\tilde{\Gamma}_\tau)\pi(A^*)\Omega) \qquad A, B \in \mathscr{B}_\tau.$$

By taking $B = A$ and using the Jensen inequality as in theorem 3.1, one obtains the EEB inequality for $A \in \mathscr{B}_\tau$. Using the closedness of δ, one extends the inequality to $\mathscr{B}_0$.

Conversely suppose that the state ω satisfies the EEB inequality, then as in the finite case it follows that ω is separating and α_t invariant. Consider again the GNS representation of the state and the corresponding operator $\tilde{\Gamma}_\tau$ on the representation space.

As above, one proves that $\tilde{\Gamma}_\tau$ is a self-adjoint operator with a dense domain of analytic vectors $\pi(\mathscr{B}_\tau)\Omega$. One is now in a position to develop the usual arguments (see, e.g., [16] § 5.3.1) to prove the EC equation.

To illustrate the type of systems and states which we consider here, we previously referred to the static Hubbard model [11]. The equilibrium states of this model are very complicated and have not been computed explicitly so far. Therefore, we describe here a soluble model and its equilibrium states which are easy, but nevertheless which do not simply factorise into classical and quantum parts. The model is a lattice of Ising spin particles in interaction with a one-mode quantised electromagnetic field. The local Hamiltonians are

$$H_\Lambda = \sum_{\langle ij\rangle} J\sigma_i\sigma_j + \sum_{i\in\Lambda} a_i^+ a_i + \lambda\sigma_i(a_i + a_i^+) \tag{16}$$

where a_i, a_i^+ are Boson creation and annihilation operators and the σ_i are Ising spins. The configuration space $\mathscr{X}$ is now $\times_{i\in\Lambda}\{0, 1\}$ and the relevant C^* algebra $\mathscr{A}$ is the CCR algebra generated by the Weyl operators

$$W(f) = \exp \mathrm{i} \sum_j (f_j a_j^+ + \bar{f}_j a_j) \qquad f \in L^2(\mathbb{Z}).$$

The Hamiltonian (16) is an unbounded operator and as such the model does not fit technically in the scheme discussed above.

We compute the equilibrium states of this system. For each configuration $x \in \mathscr{X}$, consider the *-automorphism γ_x of $\mathscr{A}$:

$$\gamma_x : a_i \to a_i - \lambda\sigma_i(x)$$

or

$$\gamma_x(W(f)) = W(f) \exp\left(-\mathrm{i}\lambda \sum_j (f_j + \bar{f}_j)\sigma_j(x)\right).$$

Hence a Weyl operator is mapped to a product of a classical observable and the Weyl operator. Then

$$\tilde{H}_\Lambda = \gamma_x(H_\Lambda(x)) = -J \sum_{\langle ij\rangle} \sigma_i(x)\sigma_j(x) + \sum_{i\in\Lambda} (a_i^+ a_i - \lambda^2)$$

is reduced to the sum of a classical and a quantum Hamiltonian. The equilibrium state η_β for the system $\tilde{H}_\Lambda$ is clearly the product state:

$$\eta_\beta(XW(f)) = \omega_\beta^{\mathrm{Ising}}(X)\omega_\beta^{\mathrm{B}}(W(j))$$

for $X \in \mathscr{C}(\mathscr{X})$, $W(f) \in \mathscr{A}$ and where $\omega_\beta^{\mathrm{Ising}}$ is the Ising equilibrium state and $\omega_\beta^{\mathrm{B}}$ the equilibrium state of the free Bose gas.

The equilibrium state ω_β of the system (16) is then

$$\begin{aligned}\omega_\beta(XW(f)) &= \eta_\beta(\gamma(XW(f))) \\ &= \eta_\beta\left(X\left(\exp\left(-\mathrm{i}\lambda\sum_j(f_j+\bar{f}_j)\sigma_j\right)\right)W(f)\right) \\ &= \omega_\beta^{\mathrm{Ising}}\left(X\left(\exp\left(-\mathrm{i}\lambda\sum_j(f_j+\bar{f}_j)\sigma_j\right)\right)\right)\omega_\beta^{\mathrm{B}}(W(f))\end{aligned}$$

and ω_β is not a product state, i.e.

$$\omega_\beta(XW(f)) \neq \omega_\beta(X)\omega_\beta(W(f))$$

because

$$\omega_\beta(X)\omega_\beta(W(f)) = \omega_\beta^{\mathrm{Ising}}(X)\omega_\beta^{\mathrm{Ising}}\left(\exp\left(-\mathrm{i}\lambda\sum_j(f_j+\bar{f}_j)\sigma_j\right)\right)\omega_\beta^{\mathrm{B}}(W(f)).$$

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